technische universität dortmund

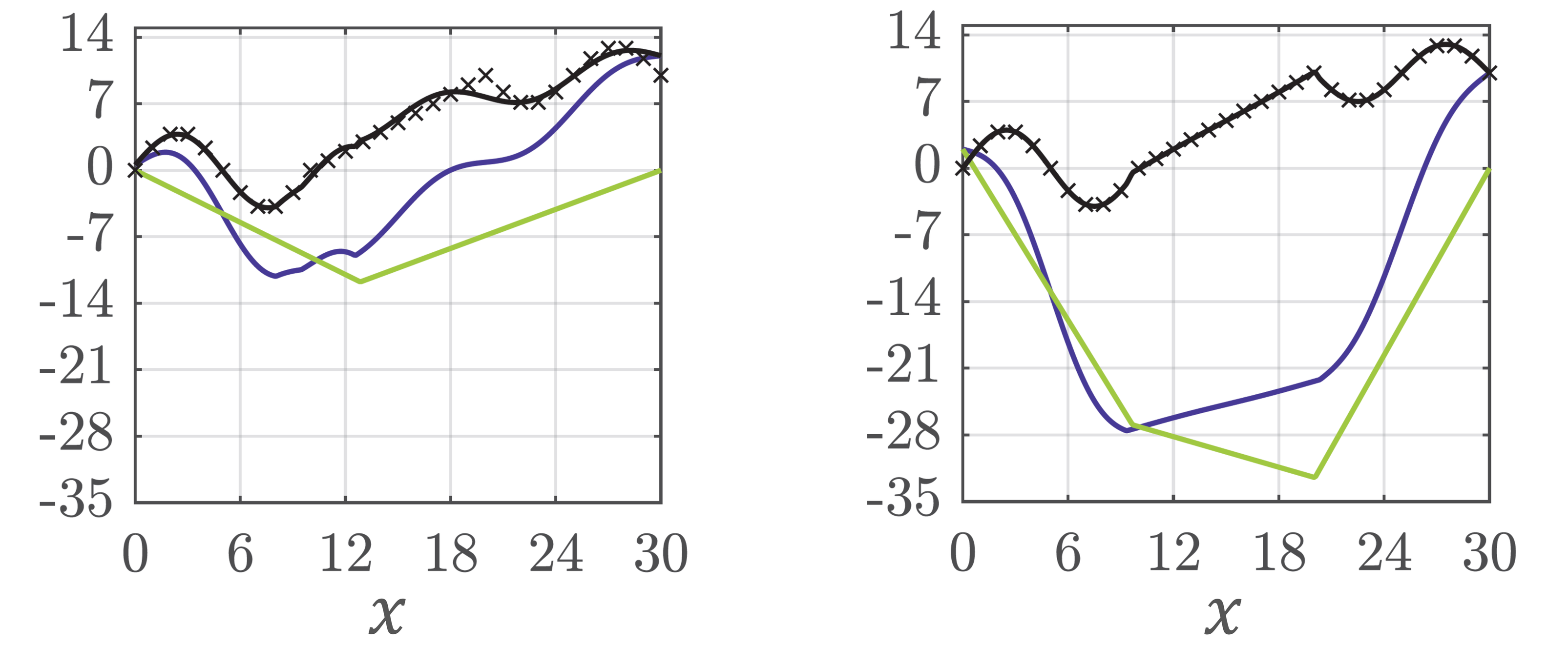
control and cyberphysical systems

Piecewise regression via mixed-integer programming for MPC

Dieter Teichrib and Moritz Schulze Darup

Motivation and Idea

- Piecewise regression can be used to approximate complex functions
- Existing approaches are often limited to a special class of functions (e.g., convex, PWA)
- Idea: Use $\Phi(\boldsymbol{x}) = \max\{\boldsymbol{V}\boldsymbol{g}(\boldsymbol{x})\} - \max\{\boldsymbol{W}\boldsymbol{h}(\boldsymbol{x})\}$ to represent arbitrary piecewise-defined functions



$\max\{\boldsymbol{V}\boldsymbol{g}(\boldsymbol{x})\}$ with $\boldsymbol{g}(\boldsymbol{x}) = (1 \quad x \quad \sin(0.2x))$

$\max\{\boldsymbol{W}\boldsymbol{h}(\boldsymbol{x})\}$ with $\boldsymbol{h}(\boldsymbol{x}) = (1 \quad x)$

$\Phi(\boldsymbol{x})$ ×: Data tuples $(\boldsymbol{x}^{(i)}, y^{(i)})$

A new MIP-based method for piecewise regression

Mixed-integer program (MIP)

- Piecewise regression can be solved by

$$\min_{V,W} \sum_{i=1}^{N_D} \left(y^{(i)} - \Phi(\boldsymbol{x}^{(i)})\right)^2 \qquad (1)$$

s.t. $\Phi(\boldsymbol{x}^{(i)}) = \max\{\boldsymbol{V}\boldsymbol{g}(\boldsymbol{x}^{(i)})\} - \max\{\boldsymbol{W}\boldsymbol{h}(\boldsymbol{x}^{(i)})\}$,

which is a non-linear optimization problem

- The mixed-integer linear constraints

$$\boldsymbol{1}\alpha^{(i)} - M(\boldsymbol{1} - \boldsymbol{\delta}^{(i)}) \leq \boldsymbol{V}\boldsymbol{g}(\boldsymbol{x}^{(i)}) \leq \boldsymbol{1}\alpha^{(i)},$$

$$\sum_{k=1}^{p_1} \delta_k^{(i)} = 1, \quad \boldsymbol{\delta}^{(i)} \in \{0,1\}^{p_1},$$

describe $\alpha^{(i)} = \max\{\boldsymbol{V}\boldsymbol{g}(\boldsymbol{x}^{(i)})\}$

- Use similar constraints to represent $\beta^{(i)} = \max\{\boldsymbol{W}\boldsymbol{h}(\boldsymbol{x}^{(i)})\}$
- Set $\Phi(\boldsymbol{x}^{(i)}) = \alpha^{(i)} - \beta^{(i)}$
- (1) becomes a mixed-integer QP

$\Rightarrow$ Can be solved using standard software

Complexity reduction

- Number of binary variables $\#_\mathrm{B}$ is
$$\#_\mathrm{B} = N_D(p_1 + p_2)$$
- Reformulation as QP for $p_1 = p_2 = N_D$
  - Substitute $\delta_i^{(i)} = 1$ for all $i \in \{1, \dots, N_D\}$
- Equivalent solutions due to permutation invariance of $\max\{x_1, x_2\} = \max\{x_2, x_1\}$
  - Symmetry breaking:
$$\boldsymbol{\delta}_{2:p_1}^{(1)} = \boldsymbol{0}, \boldsymbol{\delta}_{3:p_1}^{(2)} = \boldsymbol{0}, \dots, \boldsymbol{\delta}_{p_1}^{(p_1-1)} = \boldsymbol{0}$$
- Preclustering of data ➡ $\#_\mathrm{B} = K(p_1 + p_2)$

Benefits

- $\boldsymbol{g}(\boldsymbol{x})$ and $\boldsymbol{h}(\boldsymbol{x})$ can incorporate arbitrary continuous functions
- $\Phi(\boldsymbol{x})$ is suitable for the use in the constraints and the cost function of an OP:
  - PWA prediction model in MPC
  - Approximations of Q-functions

- Approximation of NMPC policy a) with

b) $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x}) = (1 \quad \boldsymbol{x}^\top)^\top$

c) $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x}) =$
$(1 \;\; \boldsymbol{x}^\top \;\; x_1^2 \;\; x_1x_2 \;\; x_2^2 \;\; x_1^3 \;\; x_1^2x_2 \;\; x_1x_2^2 \;\; x_2^3)^\top$